**Complex number**

**Evaluate:**

**(1)** $\left(1+i\right)^{2015}+\left(1-i\right)^{2015}$

**(2)** $1+\left(1+i\right)+\left(1+i\right)^{2}+\left(1+i\right)^{3}+…+\left(1+i\right)^{2015}$.

**Solve for all roots (including complex number roots):**

**(3)** $z^{6}+z^{3}+1=0$

**(4)** $\left(z+1\right)^{5}+\left(z-1\right)^{5}=0$ .

Beautiful fractal diagram begins with a complex number.

**(1)** **Method 1**

 Since $-i\left(1+i\right)=1-i$,

 $\left(1+i\right)^{2015}+\left(1-i\right)^{2015}=\left(1+i\right)^{2015}-i^{2015}\left(1+i\right)^{2015}$

 $=\left(1+i\right)^{2015}-i^{3}\left(1+i\right)^{2015}=\left(1+i\right)^{2015}+i\left(1+i\right)^{2015}=\left(1+i\right)\left(1+i\right)^{2015}$

 $=\left(1+i\right)^{2016}=\left(1+2i+i^{2}\right)^{1008}=\left(2i\right)^{1008}=2^{1008}$

 **Method 2** Note: $cis θ=\cos(θ+i\sin(θ))$

$\left(1+i\right)^{2015}+\left(1-i\right)^{2015}=\left[\sqrt{2}\left(cis \frac{π}{4}\right)\right]^{2015}+\left[\sqrt{2}\left(cis\left(-\frac{π}{4}\right)\right)\right]^{2015}$

 $=\sqrt{2}^{2015}\left[cis \left(\frac{π}{4}×2015\right)+cis \left(-\frac{π}{4}×2015\right)\right]$ , by de Morivre’s Theorem

 $=\sqrt{2}^{2015}\left[2 cos \left(\frac{π}{4}×2015\right)\right]=\sqrt{2}\left(2^{1007}\right)\left[2 cos \left(504π-\frac{π}{4}\right)\right]$

 $=\sqrt{2}\left(2^{1008}\right)\left[ cos \left(-\frac{π}{4}\right)\right]=\sqrt{2}\left(2^{1008}\right)\left[ \frac{1}{\sqrt{2}}\right]=2^{1008}$

 **Method 3**

 Let $x=\left(1+i\right)^{2015}+\left(1-i\right)^{2015}, y=\left(1+i\right)^{2015}-\left(1-i\right)^{2015}$

 $x+y=2\left(1+i\right)^{2015}=2\left(1+2i+i^{2}\right)^{1007}\left(1+i\right)=2\left(2i\right)^{1007}\left(1+i\right)$

 $=2^{1008}\left(-i\right)\left(1+i\right)=2^{1008}\left(1-i\right)$ … (1)

 $x-y=2\left(1-i\right)^{2015}=2\left(1-2i+i^{2}\right)^{1007}\left(1-i\right)=2\left(-2i\right)^{1007}\left(1-i\right)$

 $=-2^{1008}\left(-i\right)\left(1-i\right)=2^{1008}\left(1+i\right)$ … (2)

 $\frac{\left(1\right)+(2)}{2}, x=2^{1008} .$ We also find $\frac{\left(1\right)-(2)}{2}, y=-2^{1008}i .$

**(2)** $1+\left(1+i\right)+\left(1+i\right)^{2}+\left(1+i\right)^{3}+…+\left(1+i\right)^{2015}=\frac{\left(1+i\right)^{2016}-1}{\left(1+i\right)-1}=\frac{\left(1+2i+i^{2}\right)^{1008}-1}{i}$

 $=\left(-i\right)\left[\left(2i\right)^{1008}-1\right]=\left(-i\right)\left[2^{1008}-1\right]=\left[1-2^{1008}\right]i$

**(3) Method 1**

 Let $w=z^{3}$

 $z^{6}+z^{3}+1=0 ⇒w^{2}+w+1=0$

 By quadratic eq. formula, $w=-\frac{1}{2}\pm \frac{\sqrt{3}}{2}i=cis \left(\pm \frac{2π}{3}\right)$

 $z=w^{1/3}=\left[cis \left(\pm \frac{2π}{3}+2kπ\right)\right]^{1/3}=cis \left(\pm \frac{2π}{9}+\frac{2kπ}{3}\right)$ , where k = 0, 1, 2.

 When k = 0, $z=cos \left(\frac{2π}{9}\right)\pm i sin \left(\frac{2π}{9}\right)=0.766044443119\pm 0.6427876096865i$

 When k = 1, $z=cos \left(\frac{8π}{9}\right)+i sin \left(\frac{8π}{9}\right)=-0.9396926207859+0.3420201433257i$

 $z=cos \left(\frac{4π}{9}\right)+i sin \left(\frac{4π}{9}\right)=0.1736481776669+0.9848077530122i$

 When k = 2, $z=cos \left(\frac{14π}{9}\right)+i sin \left(\frac{14π}{9}\right)=0.1736481776669-0.9848077530122i$

 $z=cos \left(\frac{10π}{9}\right)+i sin \left(\frac{10π}{9}\right)=-0.9396926207859-0.3420201433257i$

 **Method 2**

 $z^{6}+z^{3}+1=\frac{z^{9}-1}{z^{3}-1}=0 ⇒z^{9}-1=0 and z^{3}-1\ne 0$

 For $z^{9}-1=0⇒z^{9}=1=cis 2kπ ⇒z=cis \frac{2kπ}{9}$ , k = 0, 1, 2, …, 8

 For $z^{3}-1=0⇒z^{3}=1=cis 2kπ ⇒z=cis \frac{2kπ}{3}$ , k = 0, 1, 2.

 Hence the roots are $z=cis \frac{2π}{9}, cis \frac{6π}{9}, cis \frac{8π}{9}, cis \frac{10π}{9}, cis \frac{14π}{9}, cis \frac{16π}{9}$ .

**(4)** **Method 1**

$$\left(z+1\right)^{5}+\left(z-1\right)^{5}=0 ⇒ \left(z+1\right)^{5}=-\left(z-1\right)^{5} ⇒ \left(\frac{z+1}{z-1}\right)^{5}=-1=cis\left(2kπ-π\right)$$

 $\frac{z+1}{z-1}=cis\left(\frac{2kπ-π}{5}\right)$ , k = 0, 1, 2, 3, 4.

 $z=\frac{cis\left(\frac{2kπ-π}{5}\right)+1}{cis\left(\frac{2kπ-π}{5}\right)-1}=\frac{cis\left(\frac{2kπ-π}{10}\right)\left[cis\left(\frac{2kπ-π}{10}\right)+cis\left(-\frac{2kπ-π}{10}\right)\right]}{cis\left(\frac{2kπ-π}{10}\right)\left[cis\left(\frac{2kπ-π}{10}\right)-cis\left(-\frac{2kπ-π}{10}\right)\right]}=\frac{2cos\left(\frac{2kπ-π}{10}\right)}{2isin\left(\frac{2kπ-π}{10}\right)}=-cot\left(\frac{2kπ-π}{10}\right) i$

 When k = 0, $z=-cot\left(\frac{-π}{10}\right)≈3.0776835371753i$

 When k = 1, $z=-cot\left(\frac{π}{10}\right)≈-3.0776835371753i$

 When k = 2, $z=-cot\left(\frac{3π}{10}\right)≈-0.7265425280054i$

 When k = 3, $z=-cot\left(\frac{5π}{10}\right)=0$

 When k = 4, $z=-cot\left(\frac{7π}{10}\right)≈0.7265425280054i$

 **Method 2**

 $\left(z+1\right)^{5}+\left(z-1\right)^{5}=0 ⇒z^{5}+10z^{3}+5z=0 ⇒z\left(z^{4}+10z^{2}+5\right)=0$

 Therefore $z=0$ or $z^{4}+10z^{2}+5=\left(z^{2}\right)^{2}+10z^{2}+5=0$

 $z^{2}=2 \sqrt{5}-5 or z=-2 \sqrt{5}-5$

 $z=\pm \sqrt{2 \sqrt{5}-5} or \pm \sqrt{-2 \sqrt{5}-5}$

 $z≈\pm 0.7265425280054i or \pm 3.0776835371753i$

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